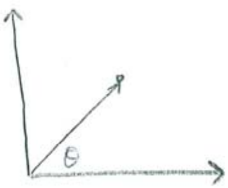
2.003 2008 12P
Final

$\dot{\theta} = \omega \quad \alpha = \dot{\omega} = \ddot{\theta}$

$\ddot{\theta} = \alpha$



$\hat{e}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$

$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$

$\hat{e}_\theta = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$

$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$

LMB: Linear Momentum Balance

$\sum F = \frac{d}{dt} m\underline{V}$

$-T\hat{e}_r + mg\hat{\imath} = m\frac{d}{dt}(l\dot{\hat{e}}_r)$

$= ml\ddot{\theta}\hat{e}_\theta + ml\dot{\theta}\dot{\hat{e}}_\theta$

$-T\hat{e}_r + mg\hat{\imath} = ml\ddot{\theta}\hat{e}_\theta - ml\dot{\theta}^2\hat{e}_r$

$-mg\sin\theta = ml\ddot{\theta}$

AMB: Angular momentum balance.

$\underline{\tau}_o = \frac{d}{dt} h_o$

$r_o \times \sum \underline{F} = \frac{d}{dt}(\underline{r}_o \times m\underline{V})$

$l\hat{e}_r \times (-T\hat{e}_r + mg\hat{\imath}) = m\frac{d}{dt}(l\hat{e}_r + l\dot{\theta}\hat{e}_\theta)$

Energy

$KE + PE = \text{const}$

$\frac{1}{2}mv^2 + mgh = \text{Const}$

$\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta = \text{const}$

$V = l\dot{\theta}\hat{e}_\theta$

$h = -l\cos\theta$

Process

1) Coordinates
2) Write position vectors
3) take derivatives
4) $F = ma$

$\delta W^{NC} = \Xi_1\delta x_1 + \Xi_2\delta x_2$

Linear Momentum

$\sum F = \frac{d}{dt}(m\underline{V})$

Angular Momentum

$h_B = \underline{r} \times m\underline{V}$

$\dot{h}_B = \dot{\underline{r}} \times m\underline{V} + r \times m\dot{\underline{V}}$

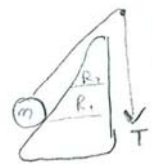
KE

$\frac{1}{2}mV^2$

$\frac{1}{2}I_c\omega^2 = \text{rot KE}$

$\tau_B = \dot{h}_B + V_B \times m\underline{V}$

Work Energy Principle



$I\omega_1 = I_2\omega_2$

$W_{1\to2} = mgh + \frac{1}{2}I\omega^2$

$\underline{H}_B = \underline{H}_c + r_c' \times p$

$\tau_B^{ext} = \dot{H}_B + V_B \times \underline{p}$

$\underline{p} \to m\dot{r}_{cB}$

Do Small 4 approx @ end

Impulse

$\mathcal{F} = \Delta p\,\delta(t) \Rightarrow \int_{t=0^-}^{t=0^+} \Delta p\,\delta(t)\,dt = \Delta p = m[V(0^+) - V(0^-)]$

$\int_{0^-}^{0^+} \delta(t)\,dt = 1$

$V_B = V_A + \vec{\omega} \times r_{AB}$

Linear Momentum Conservation

$m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2'$

Coefficient of Restitution

$e = \frac{\text{rel Vel after}}{\text{rel Vel before}}$

$e = \frac{V_2' - V_1'}{V_1 - V_2}$

Rigid Body Kinematics

$\underline{V}_b = \underline{V}_a + \underline{\omega}_{ab} \times \underline{r}_{ab}$

Instant centers

Know how to points are moving. then calculate $\perp$ to intersect.

$e = 0$ masses stick together

$e = 1$ perfectly elastic case
KE conserved.

e Only affects direction of impact.

Principle of Virtual Work

$\delta W = \sum_i \mathcal{F}_i^{applied} \cdot \delta r_i = 0$

$= \sum \Xi_j \cdot \delta\xi_j$

Moments of Inertia

$I_B = I_{cm} + m r_{cm\to B}'^2$

$\frac{1}{2}mR^2$ disk

mr^2 point

$\frac{1}{12}mL^2$ rod

Lagrange's

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right) - \frac{\partial L}{\partial \xi_j} = \Xi_j$

Step 1 Degrees of Freedom
Step 2 Conserve / non conserve forces
Step 3 $L = T - V$
Step 4 Ξ_j $F_c\delta_c + F_D\delta_D + \dots \Xi_j\delta_j$ only nonconservative
Step 5 Complete Lagrange

$V = \frac{1}{2}Kx^2 + mgh \quad T = \frac{1}{2}mV^2 + \frac{1}{2}I_c\dot{\theta}^2$

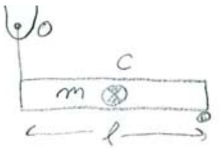
Breaking down ODE's

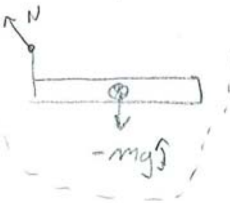
$X_1 = X \quad \dot{X}_1 = \dot{X}$

$X_2 = \dot{X}_1 = \dot{x}$

$\Rightarrow \dot{X}_2 = \ddot{X}$ & so forth

$\boxed{\Xi_j \cdot \delta_j}$





$\Sigma \underline{F} = m\, a_{cm}$ $\quad \dot{\omega} = \alpha$

$\underline{N} - mg\hat{j} = m\, \underline{\alpha} \times \underline{r}_{oc}$

$= m\ddot{\theta}\hat{k} \times (\frac{l}{2}\hat{i} - h\hat{j})$

LMB $\rightarrow$ $\Sigma F = m a_c$

AMB $\rightarrow$ $\Sigma \tau_o = I_o \alpha$

Trig

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

$$\cos(2\alpha) = 2\cos^2(\alpha) - 1 = \cos^2(\alpha) - \sin^2(\alpha)$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Derivatives

S	cos	C	−csc cot
C	−sin	S	sec tan
T	$\sec^2$	C	$-\csc^2$

$$e^{jz} = \cos z + j\sin z$$

18.03 $\quad ay'' + by' + cy = 0$

if r_1 & r_2 real $\quad y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

if $r_1 = 0$ & r_2 real $\quad y(x) = C_1 + C_2 e^{r_2 x}$

if $r_1 = r_2$ $\quad y(x) = (C_1 + C_2 x) e^{r_1 x}$

$\ddot{\alpha} + A\alpha - B = 0$

if + wiggles & stable

$\omega^2 = A$

$\alpha(t) = C_1 \cos\omega t + C_2 \sin\omega t$

if Real part s_1 & s_2 negative → stable

if Real part s_1 & s_2 positive → unstable

Real part s_1 & $s_2 = 0$ neutrally stable.

Vibrations steps

Get equations of motion

Separate into matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{\text{mass}} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_1+K_2 \end{bmatrix}}_{\text{stiffness}} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\underline{M}\ddot{X} + \underline{K}X = 0$

assume solution

$$\underline{X} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} \cos(\omega t - \phi)$$

No constant terms @ equilibrium

put back in equation

$$\begin{bmatrix} -m\omega^2 + K_1 + K_2 & -K_2 \\ -K_2 & -m\omega^2 + K_1 + K_2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Det = 0

$$\begin{matrix} \alpha & \gamma \\ \gamma & \beta \end{matrix}$$

$\alpha\beta - \gamma\gamma$

Solve for ω

$m\omega^2 = K_1$ or $m\omega^2 = K_1 + 2K_2$

multiple frequency solution

$\omega_1, \omega_2, \omega_n$

$\omega_1 \Rightarrow \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

$\omega_2 \Rightarrow \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

Equilibrium Stuff

Step 1: set all velocities & accelerations to zero. Find conditions for α's & displacement.

Step 2: Analyze small disturbances.

$\theta \ll 1$, $x \ll 1$ $\quad s = s_0 + \underset{\text{small}}{\epsilon}$

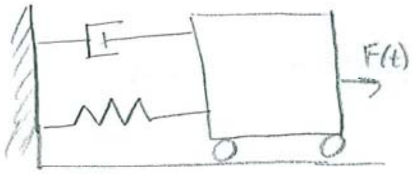
Step 3: Linearize equations of motion for small disturbances.

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\cos(\omega_1 t - \phi_1) + B\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\cos(\omega_2 t - \phi_2)$$

A, B, ϕ_1, ϕ_2 from IVP

Guessed

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} \cos(\omega t - \phi)$$



$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

$$X(t) = \underbrace{\text{Free Response}}_{\text{Complementary Solution}} + \underbrace{\text{Response due to Force}}_{\text{particular Solution}}$$

$$m\ddot{x} + c\dot{x} = 0$$

$$V = V_0 e^{-ct/m}$$

integrate to get

$$X = X_0 + \frac{mV_0}{c}\left(1 - e^{-\frac{ct}{m}}\right)$$

if $r_1 = r_2$

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

$$y'' + 2y' = 0$$

$$r^2 + 2r = r(r+2)$$

$$y(x) = c_1 + c_2 e^{-2x}$$

18.03

$$ay'' + by' + cy = 0$$

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\left\langle\ \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0\ \right\rangle$$

$$\boxed{\omega_n^2 = \frac{k}{m} \quad \text{natural frequency}}$$

$$\boxed{\zeta = \frac{c}{2m\omega_n}}$$

to solve, assume

$$X = Ae^{\lambda t}$$

$$\lambda^2 = 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

Overdamped

$\zeta > 1 \Rightarrow \lambda_1, \lambda_2$ Real negative #'s

$$X = A \pm e^{-\left(\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t\right)}$$

Critically damped

$\zeta = 1 \quad \lambda_+, \lambda_- = -\omega_n$

$$X = (A_1 + A_2 t)e^{-\omega_n t}$$

Underdamped

$$0 \leq \zeta \leq 1$$

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm i\omega_d$$

$$X = \left[Ae^{j\omega_d t} + A_2 e^{-j\omega_d t}\right]e^{-\zeta\omega_n t}$$

$$X(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$$C = \frac{A_3}{\cos\phi} \qquad \phi = \tan^{-1}\left(\frac{A_4}{A_3}\right)$$

$$A_3 = X_0$$

$$A_4 = \frac{V_0 + \zeta\omega_n X_0}{\omega_d}$$

Periodic Function $\quad F_0\cos\omega t$

$$X_p = \mathrm{Re}\{\bar{X}e^{i\omega t}\} = \mathrm{Re}\{|\bar{X}|e^{-i\phi}e^{i\omega t}\}$$

$$= |\bar{X}|\cos(\omega t - \phi)$$

$$|\bar{X}| = \frac{F_0}{k}\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

$$\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

PS3Q1 Impulse

$I = \int_{t^-}^{t^+} \vec{F} dt = MV_2 - MV_1 = \Delta p$

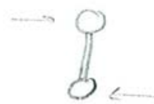
I_m = Value = $10 Ns$ $\quad m = 1.5 kg \quad l = .6m$

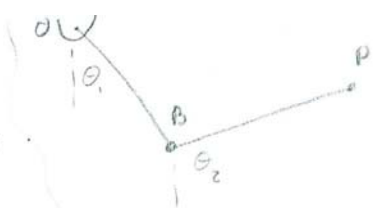
$\frac{I_m}{2m} = V_2 = 3.33 \, m/s = V_{cm}$

$I_\tau = \int_{t^-}^{t^+} r \times \vec{F} dt = \frac{l}{2} \int_{t^-}^{t^+} \vec{F} dt = \frac{l}{2} \times I_m = 3 Nms$

$\vec{\tau} = \frac{d}{dt}\left(I_{inertia} \vec{\omega}\right) = I_{iner} \omega_2 - I_{inert} \omega_1 = I_{imp\,\tau}$

$\omega_2 = \frac{I_\tau}{2 I_{inert}}$





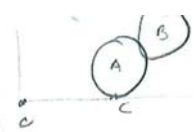
$Y_B = -L_1 \cos\theta_1$

$\delta Y_B = L_1 \sin(\theta_1) \delta\theta_1$

$Y_P = Y_B + L_2 \sin(\theta_2 - 90)$

$\delta Y_P = \delta Y_B + L_2 \cos(\theta_2 - 90) \delta\theta_2$

...



1) Find $r_{OA} = \hat{i} \quad \hat{j}$

$r_{OB} = \hat{i} \quad \hat{j}$

δr_{OA}, δr_{OB}

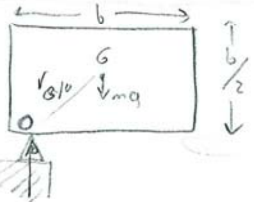
$V_B = V_A + \omega \times r_{AB}$

$V_{C=0}$

$\delta r_{OA} = \delta\theta_A \hat{k} + R\hat{j} = \delta x \hat{i}$

$\delta\theta_A = \frac{-\delta x}{A}$

PS5Q3



$\sum M_o = I_o \underline{\alpha}$

$mgx = (I_G + m|r_{GO}|^2)\alpha$

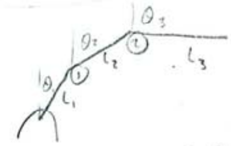
$I_g = \left(\frac{1}{12}\right) m \left(b^2 + \left(\frac{b}{2}\right)^2\right)$

$\alpha = f(x) = \frac{gx}{\frac{1}{2}b^2 + x^2}$

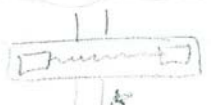
$\frac{d\alpha}{dx} = 0$

$X_{max} = b/\sqrt{b}$

$\alpha_{max} = \sqrt{3/2} \cdot g/b$



$\delta W = \underbrace{F L_1 \cos\theta_1 \delta\theta_1}_{Q_1} + \underbrace{F L_2 \cos\theta_2 \delta\theta_2}_{Q_2}$

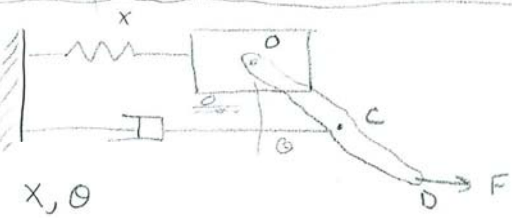


DOF 3

$\dot{x}_1, \dot{\theta}, \dot{x}_2$

position vector & take velocity

do Lagrange's method



x, θ

$V_o = \dot{x}$

$r_C = \left(x + \frac{b}{2}\sin\theta\right)\hat{i} - \frac{b}{2}\cos\theta\,\hat{j}$

$T = \frac{1}{2}MV_o^2 + \frac{1}{2}mV_c^2$

$V = \frac{1}{2}Kx^2 + mg\left(-\frac{b}{2}\cos\theta\right)$

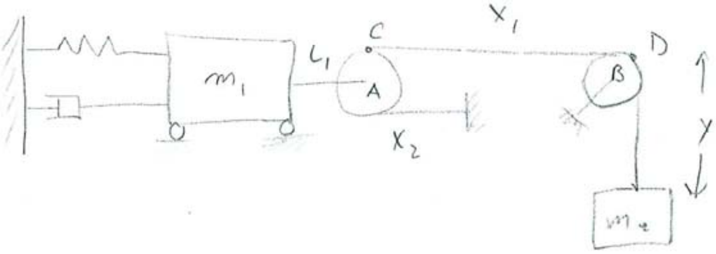
$\dot{r}_c = \dot{x}^2 + \left(\frac{b}{2}\right)^2 \dot{\theta}^2 \cos^2\theta + \dot{x} b \dot{\theta} \cos\theta + \frac{b^2}{4}\sin^2\theta\,\dot{\theta}^2$

$V_c^2 = \dot{x}^2 + \frac{b^2}{4}\dot{\theta}^2 + \dot{x} b \dot{\theta}\cos\theta$

$\delta r_D = \delta x \hat{i} + \delta(b\sin\theta\,\hat{i} - b\cos\theta\,\hat{j})$

$F\hat{i} \cdot \delta r_D = F\delta x + Fb\cos\theta\,\delta\theta$

Similar for δr_C



$\delta W = \sum$ total separate into $\delta\theta$, δx

do Lagrange

$\frac{\partial L}{\partial \dot{x}} \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$ all live

$\frac{\partial L}{\partial x}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \Xi_x$

$T = \frac{1}{2}m_1 V_1^2 + \frac{1}{2}m_2 V_2^2 + \frac{1}{2}m_A V_a^2 + \frac{1}{2}I_A\omega_A^2 + \frac{1}{2}I_B\omega_B^2$

$V_1 = \dot{x}$

$V_A = \dot{x}$

$L_{tot} = X_1 + X_2 + y + L_c = \text{Const}$

$\frac{d}{dt}(L_{tot}) = 0$

$\dot{X}_1 + \dot{X}_2 + \dot{y} = 0$

$\dot{X}_1 = \dot{X}_2$

$\dot{x} = -\dot{x}_1$

$\ddot{y} = 2\dot{x}$

$T_n = 0.75$ s period of undamped oscillations

$\frac{X_1}{X_2} = 4$ ratio of two successive positive displacement amp

damping ratio $\zeta =$

Viscous damp $c =$

equivalent $k =$

$$\delta = \ln\left(\frac{X_1}{X_2}\right)$$

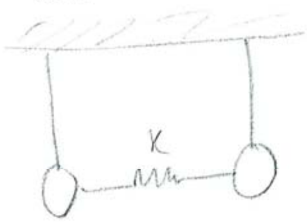
$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}$$

$$\zeta = \frac{c}{2m\omega_n}$$

$$\omega_n = \frac{2\pi}{T_n}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

Pendelum



natural fregency

$$\omega_n = \sqrt{g/L}$$

$$m\ell^2\ddot{\theta} + mg\ell\theta + 2k\ell^2\theta = 0$$

$2k$

$$\omega_n = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$